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Analytic continuation of q-Euler numbers and polynomials

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ABSTRACT

In this work we study the q-Euler numbers and polynomials analytically continued to $E_q(s)$. A new formula for Euler's q-zeta function $\zeta_{E,q}(s)$ in terms of nested series of $\zeta_{E,q}(n)$ is derived. Finally we introduce the new concept of dynamics of the zeros of analytically continued q-Euler polynomials.

1. Introduction

Throughout this work, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the ring of integers, the field of real numbers and the complex numbers, respectively.

When one talks of q-extension, q is variously considered as an indeterminate, a complex number and a p-adic number. Throughout this work, we will assume that $q \in \mathbb{C}$ with $|q| < 1$. The q-symbol $[x]_q$ denotes $[x]_q = \frac{1-q^x}{1-q}$ (see [1–16]).

In this work we study the q-Euler numbers and polynomials analytically continued to $E_q(s)$. A new formula for Euler's q-zeta function $\zeta_{E,q}(s)$ in terms of nested series of $\zeta_{E,q}(n)$ is derived. Finally we introduce the new concept of dynamics of the zeros of analytically continued q-Euler polynomials.

2. Generating q-Euler polynomials and numbers

For $h \in \mathbb{Z}$, the q-Euler polynomials were defined as

$$\sum_{n=0}^{\infty} \frac{E_n(x, h|q)}{n!} t^n = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[n+x]_q t}, \tag{2.1}$$

for $x, q \in \mathbb{C}$, cf. [1,7]. In the special case $x = 0$, $E_n(0, h|q) = E_n(h|q)$ are called q-Euler numbers; cf. [1–4]. By (2.1); we easily see that

$$E_n(x, h|q) = \frac{[2]_q}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l}{1+q^{l+h}} q^{lx}, \quad \text{cf. [7,8]}, \tag{2.2}$$

where $\binom{n}{j}$ is the binomial coefficient. From (2.1), we derive

$$E_{n,q}(x, h|q) = (q^x E(h|q) + [x]_q)^n$$

with the usual convention of replacing $E^n(h|q)$ by $E_n(h|q)$. In the case $h = 0$, $E_n(x, 0|q)$ will be symbolically written as $E_{n,q}(x)$. Let $G_q(x, t)$ be the generating function of q-Euler polynomials as follows:

$$G_q(x, t) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{2.3}$$

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Then we easily see that

$$G_q(x,t) = [2]_q \sum_{k=0}^{\infty} (-1)^k e^{[k+x]_q t}. \tag{2.4}$$

For $x = 0$, $E_{n,q} = E_{n,q}(0)$ will be called q-Euler numbers.

From (2.3), (2.4), we easily derive the following: For k (=even) and $n \in \mathbb{Z}_+$, we have

$$E_{n,q}(k) - E_{n,q} = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]_q^n. \tag{2.5}$$

For k (=odd) and $n \in \mathbb{Z}_+$, we have

$$E_{n,q}(k) + E_{n,q} = [2]_q \sum_{l=0}^{k-1} (-1)^l [l]_q^n. \tag{2.6}$$

By (2.4), we easily see that

$$E_{m,q}(x) = \sum_{l=0}^{m} \binom{m}{l} q^{xl} E_{l,q} [x]_q^{m-l}. \tag{2.7}$$

From (2.5)–(2.7), we derive

$$[2]_q \sum_{l=0}^{k-1} (-1)^{l-1} [l]_q^n = (q^{kn} - 1) E_{n,q} + \sum_{l=0}^{k-1} \binom{n}{l} q^{kl} E_{l,q} [k]_q^{n-l}, \tag{2.8}$$

where k (=even) $\in \mathbb{N}$. For k (=odd) and $n \in \mathbb{Z}_+$, we have

$$[2]_q \sum_{l=0}^{k-1} (-1)^{l} [l]_q^n = (q^{kn} + 1) E_{n,q} + \sum_{l=0}^{k-1} \binom{n}{l} q^{kl} E_{l,q} [k]_q^{n-l}. \tag{2.9}$$

3. q-Euler zeta function

It is known that the Euler polynomials are defined as

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n, \quad |t| < \pi, \quad \text{cf. [1–16]}. \tag{3.1}$$

For $s \in \mathbb{C}$, $x \in \mathbb{R}$ with $0 \le x < 1$, define

$$\zeta_E(s,x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}, \quad \text{and} \quad \zeta_E(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}. \tag{3.2}$$

Euler numbers are related to the Euler zeta function as

$$\zeta_E(-n) = E_n, \qquad \zeta_E(-n,x) = E_n(x).$$

For $s, q, h \in \mathbb{C}$ with $|q| < 1$, we define q-Euler zeta function as follows:

$$\zeta_{E,q}(s,x|h) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^{nh}}{[n+x]_q^s}, \quad \text{and} \quad \zeta_{E,q}(s|h) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^{nh}}{[n]_q^s}. \tag{3.3}$$

For $k \in \mathbb{N}$, $h \in \mathbb{Z}$, we have

$$\zeta_{E,q}(-n|h) = E_n(h|q).$$

In the special case $h = 0$, $\zeta_{E,q}(s|0)$ will be written as $\zeta_{E,q}(s)$. For $s \in \mathbb{C}$, we note that

$$\zeta_{E,q}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n}{[n]_q^s}.$$

We now consider the function $E_q(s)$ as the analytic continuation of Euler numbers. All the q-Euler numbers $E_{n,q}$ agree with $E_q(n)$, the analytic continuation of Euler numbers evaluated at n,

$$E_q(n) = E_{n,q} \quad \text{for } n \ge 0.$$

Ordinary Euler numbers are defined by

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad |t|<\pi. \tag{3.4}$$

From the definition of the Euler numbers, it is easy to show that

$$E_0=1,\quad \text{and}\quad E_n=-\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l,\quad n=0,1,2,\ldots.$$

From (4.1), we can consider the q-extension of Euler numbers E_n as follows:

$$E_{0,q}=\frac{[2]_q}{2},\quad \text{and}\quad E_{n,q}=-\frac{1}{[2]_{q^n}}\sum_{l=0}^{n-1}\binom{n}{l}q^lE_{l,q},\quad n=1,2,3,\ldots, \tag{3.5}$$

In fact, we can express $E'_q(s)$ in terms of $\zeta'_{E,q}(s)$, the derivative of $\zeta_{E,q}(s)$:

$$E_q(s)=\zeta_{E,q}(-s),\ E'_q(s)=\zeta'_{E,q}(-s),\ E'_q(2n+1)=\zeta'_{E,q}(-2n-1), \tag{3.6}$$

for $n\in\mathbb{N}\cup\{0\}$. This is just the differential of the functional equation and so verifies the consistency of $E_q(s)$ and $E'_q(s)$ with $E_{n,q}$ and $\zeta(s)$.

From the above analytic continuation of q-Euler numbers, we derive

$$E_q(s)=\zeta_{E,q}(-s),\qquad E_q(-s)=\zeta_{E,q}(s)\Rightarrow E_{-n,q}=E_q(-n)=\zeta_{E,q}(n),\ n\in\mathbb{Z}_+. \tag{3.7}$$

The curve $E_q(s)$ runs through the points $E_{-n,q}$ and grows $\sim n$ asymptotically as $(-n)\to-\infty$. The curve $E_q(s)$ runs through the point $E_q(-n)$ and $\lim_{n\to\infty}E_q(-n)=\lim_{n\to\infty}\zeta_{E,q}(n)=-2$. From these results, we note that

$$\zeta_{E,q}(-n)=E_q(n)\mapsto\zeta_{E,q}(-s)=E_q(s).$$

4. Analytic continuation of q-Euler polynomials

For consistency with the redefinition of $E_{n,q}=E_q(n)$ in (4.5) and (4.6), we have

$$E_{n,q}(x)=\sum_{k=0}^{n}\binom{n}{k}E_{k,q}q^{kx}[x]_q^{n-k}.$$

Let $\Gamma(s)$ be the gamma function. Then the analytic continuation can be obtained as

$$\begin{aligned}
&n\mapsto s\in\mathbb{R},\ x\mapsto w\in\mathbb{C},\\
&E_{k,q}\mapsto E_q(k+s-[s])=\zeta_{E,q}(-(k+(s-[s]))),\\
&\binom{n}{k}\ \mapsto\ \frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\\
&\qquad\Rightarrow\ E_{n,q}(s)\mapsto E_q(s,w)=\sum_{k=-1}^{[s]}\frac{\Gamma(1+s)E_q(k+s-[s])q^{(k+s-[s])w}[w]_q^{[s]-k}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)}\\
&\qquad=\ \sum_{k=0}^{[s]+1}\frac{\Gamma(1+s)E_q((k-1)+s-[s])q^{((k-1)+s-[s])w}[w]_q^{[s]+1-k}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)},
\end{aligned}$$

where $[s]$ gives the integer part of s, and so $s-[s]$ gives the fractional part.

Deformation of the curve $E_q(2,w)$ into the curve of $E_q(3,w)$ is via the real analytic continuation $E_q(s,w), 2\le s\le 3, -0.5\le w\le 0.5$.

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